JOURNAL OF APPROXIMATION THEORY 37, 69–88 (1983)

Determination of Chebyshev Approximations by Nonlinear Admissible Subsets

RYSZARD SMARZEWSKI

Department of Numerical Methods, M. Curie—Skłodowska University, 20-031 Lublin, Poland

Communicated by Oved Shisha

Received September 22, 1978; revised April 2, 1982

1. INTRODUCTION AND PRELIMINARIES

Let $B(X)$ denote the linear space of all bounded real-valued functions defined on a set X and let w be a prescribed function in $B(X)$ such that

$$\delta = \inf\{w(x): x \in X\} > 0. \tag{1}$$

Define a weighted uniform norm $\|\cdot\|_w$ on $B(X)$ by

$$\|f\|_w = \sup\{w(x)\,|f(x)|: x \in X\}. \tag{2}$$

In particular, when $w(x) = 1$ for all $x \in X$, the weighted uniform norm $\|\cdot\|_w$ becomes the usual uniform norm, which will be denoted by $\|\cdot\|$. Let $B(X)$ be partially ordered in the usual way by the relation $\leqslant$, i.e., let $f \leqslant g$ denote $f(x) \leqslant g(x)$ for all $x \in X$. If $f, g \in B(X)$ and $f \leqslant g$, then we denote by $[f, g]$ the closed interval in $B(X)$, i.e.,

$$[f, g] = \{h \in B(X): f \leqslant h \leqslant g\}.$$

Next, for any $f \in B(X)$ define the sets

$$Z_f = \{x \in X: f(x) = 0\}$$

and

$$M_f = \{x \in X: w(x)\,|f(x)| = \|f\|_w\}.$$

Now, let G be a nonempty proper subset of $B(X)$ and let f be a fixed element of $B(X)\setminus G$. Denote

$$\theta = \theta_f = \inf\{\|f - h\|_w: h \in G\}. \tag{3}$$

0021-9045/83/010069–20\$03.00/0

DEFINITION 1. An element $g \in G$ such that $\theta = \|f - g\|_w$ is called the best weighted approximation to f in G.

Let G_f be the set of all best weighted approximations to f in G and let N_f be the error-determining set [2], i.e., let

$$N_f = \bigcap_{g \in G_f} M_{f-g}.$$

Denote by K the set of all positive constant functions defined on X.

DEFINITION 2. The subset G of $B(X)$ is called admissible with respect to the pair (f_1, f_2), $f_1, f_2 \in B(X)$ if the following three conditions are satisfied:

(i) there exists $l \in G$, $l \geqslant f_1$, such that $g \geqslant l$ for every $g \in G$ such that $g \geqslant f_1$,

(ii) there exists $u \in G$, $u \leqslant f_2$, such that $g \leqslant u$ for every $g \in G$ such that $g \leqslant f_2$,

(iii) $g - \alpha \in G$ for every $\alpha \in K$ and $g \in G$ or $g + \alpha \in G$ for every $\alpha \in K$ and $g \in G$.

In particular, when $f_1 = f_2$ on X, we shall say that G is admissible with respect to f_1. Moreover, if G is admissible with respect to every $f \in F$ ($\varnothing \neq F \subset B(X)$), then we shall call G admissible with respect to F. Clearly, if G is admissible with respect to F, then G is admissible with respect to (f_1, f_2), for each $f_1, f_2 \in F$.

In Section 2 we shall determine the set G_f of all best weighted approximations to f by elements of an admissible subset G with respect to $(f_1, f_2) = (f - \theta/w,\ f + \theta/w)$. We shall also give explicit expressions to the error-determining set N_f and establish a nonuniqueness result for the best weighted approximations by elements of admissible subsets. In the next sections we apply the general theory of the best weighted approximation by elements of admissible subsets from Section 2 to the cases when G is equal to: the set of semi-isotone functions (Section 3); to the set of functions with the modulus of continuity bounded by a prescribed modulus of continuity (Section 4), and to the set of even functions (Section 5). These three types of the weighted approximation will also be considered in the subspace $C_b(X)$ of $B(X)$ of all bounded and continuous functions on a topological space X. Moreover, the semi-isotone weighted approximation will be considered in the space $BV(X)$ of functions of bounded variation on a chain X. We note that the results of Sections 3 and 4 develop the results obtained recently by Ubhaya in [5–7] in two distinct directions. Finally, we remark that the structures of admissible subsets can be very different. In particular, G need not be a convex or even have a weak betweenness property [4] (see Examples 2 and 3 from Section 2).

2. Approximation by Admissible Subsets

Throughout this section we shall assume that $G \subset B(X)$ is an admissible set with respect to $(f_1, f_2) = (f - \theta/w, f + \theta/w)$, where f is arbitrary fixed in $B(X)$ and $\theta = \theta_f$. Moreover, we shall denote by l and u the functions defined by (i) and (ii) in Definition 2 for the functions f_1 and f_2. At first, we prove the lemma, which will be useful in the following.

Lemma 1. *The functions l and u satisfy the inequality $l \leqslant u$.*

Proof. Note that from (ii) and (i) of Definition 2 it follows directly that

$$l \leqslant f_2 \quad \text{or} \quad u \geqslant f_1 \qquad \text{implies that} \quad l \leqslant u. \tag{4}$$

Now let us suppose, on the contrary, that l is not $\leqslant u$. Then, in view of (4), we may suppose that there exists points $s, z \in X$ such that

$$l(s) > f_2(s) \qquad \text{and} \qquad u(z) < f_1(z). \tag{5}$$

Define

$$\tilde{G} = \{ g \in G \colon g \geqslant f_1 \text{ or } g \leqslant f_2 \}.$$

Then by (i) from Definition 2 and (5) we have

$$\begin{aligned} \|f - g\|_w &\geqslant w(s)[g(s) - f(s)] \geqslant w(s)[l(s) - f(s)] \\ &> w(s)[f_2(s) - f(s)] = \theta \end{aligned}$$

for all $g \in G$, $g \geqslant f_1$. Similarly,

$$\|f - g\|_w \geqslant w(z)[f(z) - u(z)] > \theta$$

for all $g \in G$, $g \leqslant f_2$. Hence

$$\begin{aligned} &\inf\{\|f - g\|_w \colon g \in \tilde{G}\} \\ &\qquad \geqslant \min(w(s)[l(s) - f(s)],\ w(z)[f(z) - u(z)]) > \theta. \end{aligned} \tag{6}$$

We also claim that

$$\inf\{\|f - g\|_w \colon g \in G \backslash \tilde{G}\} > \theta. \tag{7}$$

Indeed, suppose that there exists a sequence $\{g_i\}$ in $G \backslash \tilde{G}$ such that $\|f - g_i\|_w \to \theta$ as $i \to \infty$. Moreover, let the first part of (iii) from Definition 2 holds, i.e., let $g - \alpha \in G$ for every $\alpha \in K$ and $g \in G$. Define

$$h_i = g_i - \alpha_i/\delta,$$

where

$$\alpha_i = \sup\{w(x)[g_i(x) - f_2(x)]: x \in X\}.$$

Since $g_i \in G \backslash \tilde{G}$ then there exists a point $t_i \in X$ such that

$$g_i(t_i) > f_2(t_i).$$

Hence $\alpha_i > 0$. Moreover,

$$\begin{aligned} w(x)[g_i(x) - f_2(x)] &= w(x)[g_i(x) - f(x) + f(x) - f_2(x)] \\ &= w(x)[g_i(x) - f(x)] - \theta \leqslant \|f - g_i\|_w - \theta \end{aligned}$$

for all $x \in X$. Hence $0 < \alpha_i \leqslant \|f - g_i\|_w - \theta$, which in turn, implies that $h_i \in G$ and that $\alpha_i \to 0$. Next, from the triangle inequality for the norm and from (1) it follows that

$$\|f - h_i\|_w \leqslant \|f - g_i\|_w + \alpha_i \|w\|/\delta \to \theta \qquad \text{as} \quad i \to \infty$$

and

$$\begin{aligned} h_i(x) = g_i(x) - \alpha_i/\delta &\leqslant g_i(x) - \alpha_i/w(x) \\ &\leqslant g_i(x) - [g_i(x) - f_2(x)] = f_2(x) \end{aligned}$$

for all $x \in X$. Hence $0 \leqslant \limsup_{i \to \infty} \|f - h_i\|_w \leqslant \theta$ and $h_i \in \tilde{G}$, which leads to a contradiction with (6). Thus, inequality (7) holds. Similarly, we may prove inequality (7) if only the second part of (iii) of Definition 2 holds. In this case, we ought to set

$$h_i = g_i + \alpha_i/\delta,$$

where

$$\alpha_i = \sup\{w(x)[f_1(x) - g_i(x)]: x \in X\}.$$

Finally, combining (6) with (7) we obtain a contradiction with the definition of θ, and so $l \leqslant u$ must hold. This completes the proof. ∎

COROLLARY 1. *$\theta \neq 0$ if and only if $f \notin G$.*

Proof. From Lemma 1 and Definition 2, it follows that

$$f - \theta/w \leqslant l \leqslant u \leqslant f + \theta/w. \tag{*}$$

If $f \in G$, then by (3) it follows immediately that $\theta = 0$. If $\theta = 0$ then by (*), we have $l = u = f$. But since $l \in G$, we conclude that $f \in G$, a contradiction. ∎

The connection between the set G_f of all best weighted approximations to f in G and elements l and u from Definition 2 for $f_1 = f - \theta/w$ and $f_2 = f + \theta/w$ is given in the following theorem.

THEOREM 1. *The best weighted approximation to f in G exists and $G_f = [l, u] \cap G$.*

Proof. If $g \in G$ and $l \leqslant g \leqslant u$, then it follows from (*) that $\|f - g\|_w \leqslant \theta$. Hence, by (3), $\|f - g\|_w = \theta$. If, on the other hand, $g \in G_f$ then $-\theta \leqslant w(g - f) \leqslant \theta$, which is equivalent to

$$f_1 = f - \theta/w \leqslant g \leqslant f + \theta/w = f_2.$$

The proof is completed. ∎

Now, we shall study the properties of the error-determining set N_f. To this purpose we shall need the following two lemmas.

LEMMA 2. *For $f \in B(X) \backslash G$ we have $Z_{f_2 - l} \cap Z_{f_1 - u} = \varnothing$.*

Proof. Let us suppose the contrary. Then, there exists $z \in X$ such that $f_2(z) - l(z) = f_1(z) - u(z) = 0$. Hence $u(z) - l(z) = f_1(z) - f_2(z) = f(z) - \theta/w(z) - [f(z) + \theta/w(z)] = -2\theta/w(z)$. It follows from $\theta > 0$ and $w(z) > 0$ that $u(z) < l(z)$, which contradicts Lemma 1. This proves the lemma. ∎

LEMMA 3. *For the functions l, u we have*

$$M_{f-l} = Z_{f_1 - l} \cup Z_{f_2 - l} \qquad \textit{and} \qquad M_{f-u} = Z_{f_1 - u} \cup Z_{f_2 - u}.$$

Proof. First, we shall prove the first equality. Let $z \in M_{f-l}$. By Theorem 1 this is equivalent to

$$f(z) - l(z) = \theta/w(z) \qquad \text{or} \qquad f(z) - l(z) = -\theta/w(z).$$

It follows from the definitions of f_1 and f_2 that these equalities are equivalent to

$$f_1(z) - l(z) = 0 \qquad \text{or} \qquad f_2(z) - l(z) = 0,$$

which in turn, is equivalent to $z \in Z_{f_1 - l} \cup Z_{f_2 - l}$. Similarly, we may show the second equality. The proof is completed. ∎

In the following theorem, as in Lemma 2, we shall assume that $f \notin G$. If $f \in G$ then obviously we have $l = u = f$ and $N_f = X$.

THEOREM 2. *If* $f \in B(X)\backslash G$ *then the error-determining set* N_f *satisfies inclusions* $D_f \subset N_f \subset D_f \cup E_f$, *where*

$$D_f = (Z_{f_1-l} \cap Z_{f_1-u}) \cup (Z_{f_2-l} \cap Z_{f_2-u}) \quad \text{and} \quad E_f = Z_{f_1-l} \cap Z_{f_2-u}.$$

Proof. From the definition of the set N_f and from $l, u \in G_f$ (see Theorem 1) it immediately follows that

$$N_f \subset M_{f-l} \cap M_{f-u}.$$

Hence by Lemmas 2 and 3 we have

$$N_f \subset M_{f-l} \cap M_{f-u} = D_f \cup E_f.$$

Thus, it is enough to prove that $D_f \subset N_f$. Let $g \in G_f = [l, u] \cap G$ be arbitrarily fixed and let $z \in D_f$. At first, suppose that $z \in Z_{f_1-l} \cap Z_{f_1-u}$. Then, $l(z) = g(z) = u(z)$, and consequently

$$f(z) - g(z) = f(z) - l(z) = f_1(z) - l(z) + \theta/w(z) = \theta/w(z).$$

From this and from Theorem 1 we conclude that $z \in M_{f-g}$ and so $z \in N_f$, since g is arbitrary in G_f. Thus, we conclude that $Z_{f_1-l} \cap Z_{f_1-u} \subset N_f$. Similarly, we may prove that $Z_{f_2-l} \cap Z_{f_2-u} \subset N_f$. The proof of the theorem is completed. ∎

Remark 1. If there exists $\lambda \in (0, 1)$ such that $g = (1-\lambda)\, l + \lambda u \in G$ then in Theorem 2 we have $N_f = D_f$. Indeed, in this case from $z \in E_f$ it follows that

$$\begin{aligned} f(z) - g(z) &= |\lambda[f(z) - u(z)] + (1-\lambda)[f(z) - l(z)]| \\ &= |(1-\lambda)\, \theta/w(z) - \theta/w(z)| \\ &= |1 - 2\lambda|\, \theta/w(z) < \theta/w(z). \end{aligned}$$

Hence $z \notin M_{f-g}$. Moreover, from $g = (1+\lambda)\, l + \lambda u \in G_f$ we conclude that $z \notin N_f$. This and Theorem 2 imply that $N_f = D_f$. In particular, $N_f = D_f$ when G is a convex set. Analogously, we may prove that the equality $N_f = D_f$ also holds for the sets having the betweenness property [2]. But the converse statement—$N_f = D_f$ implies that the set G has the betweenness property—is not true (see Example 5 below).

Now we shall give three examples which show that both inclusions in Theorem 2 can be neither improved nor replaced by the equalities $N_f = D_f$ or $N_f = D_f \cup E_f$ in the case of an arbitrarily admissible set G.

EXAMPLE 1. Let $w(x)=1$ and $f(x)=x^2$ for all $x \in [-1,1]$ and let G be a set of all nondecreasing functions in $C[-1,1]$. Then $\theta=\frac{1}{2}$, G is admissible with respect to the pair $(f_1(x), f_2(x))=(x^2-\frac{1}{2}, x^2+\frac{1}{2})$, $l(x)=\frac{1}{2}$ and $u(x)=\frac{1}{2}+[\max(0,x)]^2$. Moreover, $Z_{f_1-l}=\{-1,1\}$, $Z_{f_1-u}=\{-1\}$, $Z_{f_2-l}=\{0\}$ and $Z_{f_2-u}=[0,1]$. Hence $D_f=\{-1,0\}$ and $E_f=\{1\}$. Clearly, $\{1\} \notin M_{f-g}$, where $g \in [l,u] \cap G$ is defined by

$$\begin{aligned} g(x) &= \tfrac{1}{2}, \qquad x \in [-1, \tfrac{1}{2}] \\ &= x, \qquad x \in (\tfrac{1}{2}, 1]. \end{aligned}$$

Hence $N_f=\{-1,0\}$. Thus we have shown that in this case we have

$$\{-1,0\}=D_f=N_f \subsetneq D_f \cup E_f=\{-1,0,1\}.$$

EXAMPLE 2. Let w be identically equal to 1 and let $f(x)=|x|$ and let the admissible subset G of $C[-1,1]$ with respect to $(f_1(x), f_2(x))=(|x|-\frac{1}{2}, |x|+\frac{1}{2})$ be defined by

$$G=\{g\colon g(x)=a+\sigma\max(0,x),\ a \in \mathbb{R},\ \sigma=0 \text{ and } 1\}.$$

Then $\theta=\frac{1}{2}$, $l(x)=\frac{1}{2}$ and $u(x)=\frac{1}{2}+\max(0,x)$. Additionally, $Z_{f_1-l}=\{-1,1\}$, $Z_{f_1-u}=\{-1\}$, $Z_{f_2-l}=\{0\}$, $Z_{f_2-u}=[0,1]$, $D_f=\{-1,0\}$ and $E_f=\{1\}$. We notice that in this case we have $G \cap [l,u]=\{l,u\}$, and so $N_f=\{-1,0,1\}$. Hence

$$\{-1,0\}=D_f \subsetneq N_f=D_f \cup E_f=\{-1,0,1\}.$$

Note that in this example the set G is not convex. Moreover, this set does not have the betweenness property [2], nor does it have the weak betweenness property [4].

EXAMPLE 3. Let $w(x)=1$ and let

$$\begin{aligned} f(x) &= (x-1)^2, \qquad x \in [-2,0] \\ &= |x-1|, \qquad x \in (0,2]. \end{aligned}$$

Define the admissible subset G of $C[-2,2]$ with respect to $(f_1, f_2)=(f-\frac{1}{2}, f+\frac{1}{2})$ by

$$\begin{aligned} G=\{g\colon g(x) &= a+b[\min(0,1+x)]^2+\sigma\max(0,x-1), \\ & a, b \in \mathbb{R},\ \sigma=0 \text{ and } 1\}. \end{aligned}$$

Then $\theta = \frac{1}{2}$, $l(x) = \frac{1}{2}$ and

$$\begin{aligned} u(x) &= f_1(x), & x &\in [-2, -1] \\ &= \tfrac{1}{2}, & x &\in (-1, 1] \\ &= f_2(x), & x &\in (1, 2]. \end{aligned}$$

Hence we conclude that $E_f = \{-2, 2\}$ and

$$\{-1, 0, 1\} = D_f \subsetneq N_f = \{-2, -1, 0, 1\} \subsetneq D_f \cup E_f = \{-2, -1, 0, 1, 2\}.$$

Finally, we state a corollary which gives a condition on nonuniqueness of the best weighted approximation to f in G.

COROLLARY 2. *For $f \in B(X) \backslash G$ we have $\|u - l\|_w \leqslant 2\theta$. Moreover, if $E_f \neq \varnothing$ then the best weighted approximation to f in G is nonunique and $\|u - l\|_w = 2\theta$.*

The proof of this corollary directly follows from $f_1 \leqslant l \leqslant u \leqslant f_2$, $\|f_1 - f_2\|_w = 2\theta$, and from the definition of E_f. We finish this section by giving two examples, which show that Corollary 2 is in some sense the best possible. If $E_f = \varnothing$, then the best weighted approximation to f in G can either be unique or nonunique.

EXAMPLE 4. Let $w(x) = 1$ and $f(x) = x^2$ and let G be equal to the set K of all constant functions in $C[-1, 1]$. Then, as can easily be shown: $\theta = \frac{1}{2}$, $l(x) = u(x) = \frac{1}{2}$ on $[-1, 1]$, $N_f = D_f = \{-1, 0, 1\}$ and $E_f = \varnothing$. Therefore, by Theorem 1 the best approximation to f in G is unique.

EXAMPLE 5. *Let* $w(x) = 1$, $f(x) = 0$ and let G be defined by $G = P \cup H \subset C[-1, 1]$, where

$$P = \{p\colon p(x) = |x| + a,\ a \in \mathbb{R}\} \quad \text{and} \quad H = \{h\colon h(x) = x^2 + a,\ a \in \mathbb{R}\}.$$

Obviously, the set G is admissible with respect to $(f_1, f_2) = (f - \frac{1}{2}, f + \frac{1}{2})$. Moreover, $\theta = \frac{1}{2}$, $l(x) = x^2 - \frac{1}{2}$, $u(x) = |x| - \frac{1}{2}$, $N_f = D_f = \{-1, 0, 1\}$ and $E_f = \varnothing$. In this case, by Theorem 1, two distinct elements l and u belong to G_f and thus the best approximation to f is not unique. Additionally, we have

$$\tfrac{1}{4} = \|u - l\|_w < 2\theta = 1.$$

3. Semi-Isotone Approximation

In this section we assume that X is a partially ordered set with a partial order $\leqslant$. For any element $x \in X$ define the subsets L_x and U_x of X, respectively, by

$$L_x = \{z \in X: z \leqslant x\} \qquad \text{and} \qquad U_x = \{z \in X: x \leqslant z\}.$$

Moreover, let $T = \{(x, y) \in X \times X: x \leqslant y\}$ and $t_+ = \max(0, t)$, $t \in \mathbb{R}$. Given a function s in $B(X)$ such that $s(x) \geqslant 0$ for all $x \in X$, define a subset P_s of $B(X)$ of semi-isotone functions by

$$P_s = \{g \in B(X): x \leqslant y \text{ implies } g(x) \leqslant g(y) + s(x)\}.$$

If $s = 0$ on X, then P_s coincides with the set of all isotone functions [5] on X.

Lemma 4. *For every $r \in B(X)$ the set P_s is admissible with respect to r. Moreover, l and u from Definition 2 are equal to*

$$l(x) = r(x) + \sup_{z \in L_x} [r(z) - r(x) - s(z)]_+ \tag{8}$$

and

$$u(x) = r(x) - \sup_{z \in U_x} [r(x) - r(z) - s(x)]_+ \tag{9}$$

for all $x \in X$.

Proof. Let r be arbitrarily fixed in $B(X)$ and let l and u be defined by (8) and (9), respectively. We first show that condition (i) in Definition 2 is satisfied for P_s. To this purpose, let $x \in X$ and $g \in P_s$ such that $g \geqslant r$ be arbitrarily fixed and let $y \in X$ be such that $x \leqslant y$. Obviously, $l \in B(X)$ and $l \geqslant r$. Now, we distinguish between two cases. First, if

$$r(z) \leqslant r(x) + s(z) \qquad \text{for each} \quad z \in L_x,$$

then

$$\begin{aligned} l(x) = r(x) &= r(y) + s(x) + [r(x) - r(y) - s(x)] \\ &\leqslant s(x) + r(y) + [r(x) - r(y) - s(x)]_+ \leqslant s(x) + l(y) \end{aligned}$$

and

$$l(x) = r(x) \leqslant g(x).$$

Otherwise, for every $\varepsilon > 0$ there exists $t \in L_x$ such that

$$0 < \sup_{z \in L_x} [r(z) - r(x) - s(z)]_+ \leqslant r(t) - r(x) - s(t) + \varepsilon.$$

Then

$$\begin{aligned} l(x) &\leqslant r(x) + r(t) - r(x) - s(t) + \varepsilon \\ &= \varepsilon + r(y) + [r(t) - r(y) - s(t)] \leqslant \varepsilon + l(y) \end{aligned}$$

and

$$\begin{aligned} l(x) &\leqslant r(t) - s(t) + \varepsilon \leqslant g(t) - s(t) + \varepsilon \\ &\leqslant g(x) + s(t) - s(t) + \varepsilon = g(x) + \varepsilon. \end{aligned}$$

Hence $l(x) \leqslant l(y) + s(x)$ and $l(x) \leqslant g(x)$, since ε is arbitrary. Combining these both cases, we conclude that $l \in P_s$ and $l \leqslant g$. This completes the verification of condition (i). Similarly, we may show that condition (ii) in Definition 2 holds for P_s. Obviously, condition (iii) from Definition 2 is true for P_s. Thus, the proof of the lemma is completed. ∎

We now prove

THEOREM 3. *Let $f \in B(X) \backslash P_s$. Then there exists a best weighted approximation to f in P_s, the set of all best weighted approximations to f in P_s is equal to $[l, u] \cap P_s$, and the error $\theta_f = \inf_{g \in P_s} \|f - g\|_w$ is equal to θ, where*

$$\begin{aligned} \theta &= \sup_{(x, y) \in T} \frac{w(x)\, w(y)}{w(x) + w(y)} [f(x) - f(y) - s(x)]_+, \\ l(x) &= f_1(x) + \sup_{z \in L_x} [f_1(z) - f_1(x) - s(z)]_+, \\ u(x) &= f_2(x) - \sup_{z \in U_x} [f_2(x) - f_2(z) - s(x)]_+, \\ f_1 &= f - \theta/w \qquad \text{and} \qquad f_2 = f + \theta/w. \end{aligned} \tag{10}$$

Moreover, the error determining set is equal to

$$N_f = (Z_{f_1 - l} \cap Z_{f_1 - u}) \cup (Z_{f_2 - l} \cap Z_{f_2 - u}).$$

In the case when $Z_{f_1 - l} \cap Z_{f_2 - u} \neq \varnothing$, a best weighted approximation to f in P_s is nonunique and $\|u - l\|_w = 2\theta$.

Proof. By Theorems 1 and 2, Remark 1, Corollary 2 and Lemma 4 it is sufficient to prove that θ_f is equal to θ defined in (10). Let us suppose that

$g \in P_s$ is arbitrarily fixed. Then $s(x) + g(y) - g(x) \geqslant 0$ for all $(x, y) \in T$. Hence

$$f(x) - f(y) - s(x) \leqslant f(x) - f(y) - s(x) + g(y) - g(x)$$
$$\leqslant \|f - g\|_w (1/w(x) + 1/w(y))$$

for all $(x, y) \in T$. This gives $\theta_f \geqslant \theta$. Thus the proof of the theorem will be completed if we show that $\|f - l\|_w \leqslant \theta$ (i.e. that $\theta_f \leqslant \theta$) holds for l defined in (10). Note that

$$w(x)[f(x - l(x)] \leqslant w(x)[f(x) - f_1(x)] = \theta$$

for all $x \in X$. Hence it is sufficient to prove that

$$w(x)[l(x) - f(x)] \leqslant \theta \tag{11}$$

for all $x \in X$. First, let us suppose that $x \in X$ is such that

$$f_1(z) \leqslant f_1(x) + s(z) \qquad \text{for all} \quad z \in L_x.$$

Then

$$w(x)[l(x) - f(x)] = w(x)[f_1(x) - f(x)] = -\theta \leqslant \theta,$$

i.e., inequality (11) holds in this case. Otherwise, for every $\varepsilon > 0$ there exists $t \in L_x$ such that

$$0 < \sup_{z \in L_x} [f_1(z) - f_1(x) - s(z)]_+ \leqslant f_1(t) - f_1(x) - s(t) + \varepsilon.$$

Then, by definitions of l, f_1 and θ given in (10), we obtain

$$\begin{aligned}
&w(x)[l(x) - f(x)] \\
&\quad \leqslant w(x)[f_1(x) + f_1(t) - f_1(x) - s(t) + \varepsilon - f(x)] \\
&\quad = w(x)[f(t) - f(x) - s(t) - \theta/w(t) + \varepsilon] \\
&\quad \leqslant w(x) \left[f(t) - f(x) - s(t) - \frac{w(x)}{w(t) + w(x)} (f(t) - f(x) - s(t)) + \varepsilon\right] \\
&\quad = w(x) \left[\frac{w(t)}{w(t) + w(x)} (f(t) - f(x) - s(t)) + \varepsilon\right] \\
&\quad \leqslant \frac{w(t)\, w(x)}{w(t) + w(x)} [f(t) - f(x) - s(t)]_+ + \varepsilon \|w\| \leqslant \theta + \varepsilon \|w\|.
\end{aligned}$$

Since ε is arbitrary, then inequality (11) is also satisfied in this case. Thus $\|f-l\|_w \leqslant \theta$. This completes the proof. ∎

The fact that a best weighted approximation to f in P_s may be nonunique is considered in Example 1 of Section 2. The best weighted approximation may also be unique. Indeed, if $w=1$ and $s=0$ on $[-1, 1]$, $G=P_s \subset B[-1, 1]$, and $f(x)=\min(0, -x)$ then $f_1=-1$ and $f_2=0$ on $[-1, 1]$. Hence $l(x)=u(x)=-\frac{1}{2}$ on $[-1, 1]$, and so the best approximation is unique.

From now to the end of this section X will be a chain. It is well known (see, e.g. [1, p. 39]) that a chain X is a normal Hausdorff space under its intrinsic topology generated by the family of open intervals in X. In the following we assume that X is endowed with this topology. Denote by $C_b(X)$ the space of all real-bounded and continuous functions defined on a chain X. Moreover, let $BV(X)$ be the space of all functions of bounded variation [1, p. 74] on a chain X. In the investigation of the semi-isotone approximation in the spaces $C_b(X)$ and $BV(X)$ the following two lemmas are of importance.

LEMMA 5. *Let $r, v \in C_b(X)$. Then the functions h and p defined by*

$$h(x)=\sup_{z \in L_x} [r(z)-v(x)]_+ \qquad \textit{and} \qquad p(x)=\sup_{z \in U_x} [r(x)-v(z)]_+$$

belong to $C_b(X)$.

Proof. Obviously, the function h and p are bounded on X. Denote by O_x the open interval in X containing x such that

$$|v(x)-v(z)| \leqslant \varepsilon/3 \qquad \text{and} \qquad |r(x)-r(z)| \leqslant \varepsilon/3 \tag{12}$$

for every $z \in O_x$ where $\varepsilon > 0$ and $x \in X$ are arbitrarily fixed. The existence of the interval O_x follows from the continuity of v and r on the chain X. Now, let y be an arbitrary element in O_x. By virtue of the definition of h there exist elements $t_x \in L_x$ and $t_y \in L_y$ depending on $\varepsilon/3$ such that

$$h(x) \leqslant \varepsilon/3 + [r(t_x)-v(x)]_+ \qquad \text{and} \qquad h(y) \leqslant \varepsilon/3 + [r(t_y)-v(y)]_+. \tag{13}$$

Define

$$t_{xy} = \begin{cases} t_x, & \text{if } t_x \leqslant y \\ y, & \text{otherwise} \end{cases} \qquad \text{and} \qquad t_{yx} = \begin{cases} t_y, & \text{if } t_y \leqslant x \\ x, & \text{otherwise.} \end{cases}$$

Note that from the definitions of t_x and t_{xy} it follows that the equality $t_{xy}=y$

implies that $t_x \in O_x$. Thus by the definition of h, (12), (13) and the fact that $t_{xy} \in L_y$ we have

$$\begin{aligned} h(x) - h(y) &\leqslant \varepsilon/3 + [r(t_x) - v(x)]_+ - [r(t_{xy}) - v(y)]_+ \\ &= \varepsilon/3 + \tfrac{1}{2}(v(y) - v(x) + r(t_x) - r(t_{xy}) \\ &\quad + |r(t_x) - v(x)| - |r(t_{xy}) - v(y)|) \\ &\leqslant \varepsilon/3 + |v(y) - v(x)| + |r(t_x) - r(t_{xy})| \leqslant \varepsilon. \end{aligned}$$

Similarly, we show that

$$\begin{aligned} h(y) - h(x) &\leqslant \varepsilon/3 + [r(t_y) - v(y)]_+ - [r(t_{yx}) - v(x)]_+ \\ &\leqslant \varepsilon/3 + |v(x) - v(y)| + |r(t_y) - r(t_{yx})| \leqslant \varepsilon. \end{aligned}$$

Combining these both inequalities, we conclude that $h \in C_b(X)$. In a similar manner we may show that $p \in C_b(X)$. The proof is completed. ∎

LEMMA 6. *If $r, v \in BV(X)$ and the functions h, p are defined as in Lemma 5 then $h, p \in BV(X)$.*

Proof. Since $r, v \in BV(X)$ then there exists a constant $c > 0$ such that for every finite chain $z_0 < z_1 < \cdots < z_n$ in X we have

$$\sum_{i=1}^{n} |w(z_i) - w(z_{i-1})| < c, \tag{14}$$

where $w = r$ or $w = v$. Now, let $\varepsilon > 0$ be arbitrary and let $x_0 < x_1 < \cdots < x_n$ be a finite chain in X. From the definition of p it follows that there exists $t_i \in U_{x_i}$ such that

$$p(x_i) \leqslant \varepsilon/n + [r(x_i) - v(t_i)]_+$$

for $i = 0, 1, \ldots, n$. From this and from $t_i \in U_{x_i} \subset U_{x_{i-1}}$ we have

$$\begin{aligned} p(x_i) - p(x_{i-1}) &\leqslant \varepsilon/n + [r(x_i) - v(t_i)]_+ - [r(x_{i-1}) - v(t_i)]_+ \\ &= \varepsilon/n + \tfrac{1}{2}(r(x_i) - v(t_i) - r(x_{i-1}) + v(t_i) \\ &\quad + |r(x_i) - v(t_i)| - |r(x_{i-1}) - v(t_i)|) \\ &\leqslant \varepsilon/n + \tfrac{1}{2}(r(x_i) - r(x_{i-1}) + |r(x_i) - r(x_{i-1})|) \\ &\leqslant \varepsilon/n + |r(x_i) - r(x_{i-1})| \end{aligned} \tag{15}$$

for $i = 1, 2,.., n$. Moreover, denoting

$$y_i = t_{i-1}, \qquad \text{if} \quad x_i \leqslant t_{i-1}$$
$$= x_i, \qquad \text{otherwise}$$

and noting that $y_i \in U_{x_i}$ we obtain

$$\begin{aligned} p(x_{i-1}) - p(x_i) \leqslant \varepsilon/n + \tfrac{1}{2}(r(x_{i-1}) - r(x_i) + v(y_i) - v(t_{i-1}) \\ + |r(x_{i-1}) - v(t_{i-1})| - |r(x_i) - v(y_i)|) \\ \leqslant \varepsilon/n + |r(x_i) - r(x_{i-1})| + |v(y_i) - v(t_{i-1})| \end{aligned} \tag{16}$$

for $i = 1, 2,..., n$. Now, let $I_1 = \{i\colon\ y_i = t_{i-1},\ \ i = 1, 2,..., n\}$ and $I_2 = \{1, 2,..., n\} \backslash I_1 = (i_1, i_2,..., i_k)$, where $i_j < i_k$ for $j < k$. Then, in view of (15) and (16), we get

$$|p(x_i) - p(x_{i-1})| \leqslant \varepsilon/n + |r(x_i) - r(x_{i-1})|$$

for all $i \in I_1$ and

$$|p(x_i) - p(x_{i-1})| \leqslant \varepsilon/n + |r(x_i) - r(x_{i-1})| + |v(x_i) - v(t_{i-1})|$$

for all $i \in I_2$. Hence

$$\begin{aligned} \sum_{i=1}^{n} |p(x_i) - p(x_{i-1})| \\ \leqslant \varepsilon + \sum_{i=1}^{n} |r(x_i) - r(x_{i-1})| + \sum_{i \in I_2} |v(x_i) - v(t_{i-1})|. \end{aligned}$$

Since $x_{i_1-1} \leqslant t_{i_1-1} < x_{i_1} \leqslant x_{i_2} \leqslant \cdots \leqslant x_{i_k-1} \leqslant t_{i_k-1} < x_{i_k}$ then from (14) we conclude that

$$\sum_{i=1}^{n} |p(x_i) - p(x_{i-1})| \leqslant 2c,$$

i.e., that $p \in BV(X)$. The proof of $h \in BV(X)$ is similar. This completes the proof. ▮

From Theorem 3 and Lemmas 5 and 6 we obtain the following theorem.

THEOREM 4. *Let* $w, s \in C_b(X)$ *and* $G = P_s \cap C_b(X)$ ($w, s \in BV(X)$ *and* $G = P_s \cap BV(X)$)*. Then, for each* $f \in C_b(X) \backslash G$ ($f \in BV(X) \backslash G$) *there exists a best weighted approximation to* f *in* G*, the set of all weighted approximations* G_f *is equal to* $[l, u] \cap G$*, and the error* θ_f *is equal to* θ*, where* $l, u \in G_f$ *and* θ *are given by formulae* (10) *from Theorem* 3*. Moreover, the error determining* *set is given by the same formula as in Theorem* 3, $\|u-l\|_w \leqslant 2\theta$, *and* $E_f \neq \varnothing$ *implies that* $\|u-l\|_w = 2\theta$.

4. Approximation by Subsets with Prescribed Modulus of Continuity

Let $\omega \in B[0, \infty)$ be a function satisfying

$$0 \leqslant \omega(y) - \omega(x) \leqslant \omega(y-x)$$

whenever $0 \leqslant x \leqslant y$. In this section we assume that X is a subset of the real line. Define

$$H_\omega = \{g \in B(X) \colon |g(x) - g(y)| \leqslant \omega(|x-y|) \text{ for each } x, y \in X\}.$$

Now, we shall study best weighted approximations of any $f \in B(X) \backslash H_\omega$ by elements of H_ω. To this purpose the following lemma will be needed.

LEMMA 7. *The set* H_ω *is admissible with respect to each* $r \in B(X)$. *Moreover,* l *and* u *from Definition* 2 *are equal to*

$$l(x) = r(x) + \sup_{z \in X} [r(z) - r(x) - \omega(|x-z|)]_+ \tag{17}$$

and

$$u(x) = r(x) - \sup_{z \in X} [r(x) - r(z) - \omega(|x-z|)]_+. \tag{18}$$

Proof. We prove that condition (ii) from Definition 2 is satisfied for H_ω and for u given in (18); the proof of (i) is similar and therefore is omitted. To this purpose let $y \in X$ and $g \in H_\omega$ be such that $x \leqslant y$. It is obvious that $u \in B(X)$ and $u \leqslant r$. We complete the proof of (ii) showing that $g(y) \leqslant u(y)$, $u(y) \geqslant u(x) - \omega(y-x)$ and $u(x) \geqslant u(y) - \omega(y-x)$, i.e., $g \leqslant u$ and $u \in H_\omega$. At first suppose that $y \in X$ is such that

$$r(y) - r(z) \leqslant \omega(|y-z|).$$

Then by the definition of u we have

$$\begin{aligned} u(y) = r(y) &= r(x) - \omega(y-x) - [r(x) - r(y) - \omega(y-x)] \\ &\geqslant r(x) - \omega(y-x) - [r(x) - r(y) - \omega(y-x)]_+ \\ &\geqslant u(x) - \omega(y-x) \end{aligned} \tag{19}$$

and

$$u(y) = r(y) \geqslant g(y). \tag{20}$$

Otherwise, for every $\varepsilon > 0$ there exists $t \in X$ such that

$$0 < \sup_{z \in X} [r(y) - r(z) - \omega(|y-z|)]_+ \leqslant r(y) - r(t) - \omega(|y-t|) + \varepsilon.$$

Then, by the definitions of u, ω and H_ω, we easily deduce

$$\begin{aligned} u(y) &\geqslant r(x) - r(x) + r(t) + \omega(|y-t|) - \varepsilon \\ &= r(x) - [r(x) - r(t) - \omega(|t-x|)] + \omega(|y-t|) - \omega(|t-x|) - \varepsilon \\ &\geqslant r(x) - [r(x) - r(t) - \omega(|t-x|)]_+ - \omega(y-x) - \varepsilon \\ &\geqslant u(x) - \omega(y-x) - \varepsilon \end{aligned} \tag{21}$$

and

$$u(y) \geqslant r(t) - \omega(|y-t|) - \varepsilon \geqslant g(t) + \omega(|y-t|) - \varepsilon \geqslant g(y) - \varepsilon. \tag{22}$$

On the other hand, if x is such that $r(x) - r(z) \leqslant \omega(|x-z|)$ for all $z \in X$ then

$$\begin{aligned} u(x) = r(x) &\geqslant r(y) - [r(y) - r(x) - \omega(y-x)]_+ - \omega(y-x) \\ &\geqslant u(y) - \omega(y-x). \end{aligned} \tag{23}$$

In the opposite case, for every $\varepsilon > 0$ there exists $t \in X$ such that $u(x) \geqslant r(t) + \omega(|x-t|) - \varepsilon$. Hence

$$\begin{aligned} u(x) &\geqslant r(y) - [r(y) - r(t) - \omega(|t-y|)]_+ + \omega(|x-t|) - \omega(|t-y|) - \varepsilon \\ &\geqslant u(y) - \omega(y-x) - \varepsilon. \end{aligned} \tag{24}$$

Since ε is arbitrary, then by (19)–(24) it follows that $u \geqslant g$ and $u \in H_\omega$. This completes the proof of (ii). Since the verification of condition (iii) in Definition 2 is trivial for H_ω, then the proof of the lemma is finished. ∎

THEOREM 5. *Let $f \in B(X) \backslash H_\omega$. Then there exists a best weighted approximation to f in H_ω, the set of all best weighted approximations to f in H_ω is equal to $[l, u] \cap H_\omega$, and the error is equal to θ, where*

$$\theta = \sup_{(x,y) \in X \times X} \frac{w(x)\, w(y)}{w(x) + w(y)} [f(x) - f(y) - \omega(|x-y|)]_+$$

$$\begin{aligned} l(x) &= f_1(x) + \sup_{z \in X} [f_1(z) - f_1(x) - \omega(|x-z|)]_+, \\ u(x) &= f_2(x) - \sup_{z \in X} [f_2(x) - f_2(z) - \omega(|x-z|)]_+, \end{aligned} \tag{25}$$

$$f_1 = f - \theta/w \qquad \textit{and} \qquad f_2 = f + \theta/w.$$

Moreover, the error determining set is equal to

$$N_f = (Z_{f_1 - l} \cap Z_{f_1 - u}) \cup (Z_{f_2 - l} \cap Z_{f_2 - u}).$$

In the case when $Z_{f_1 - l} \cap Z_{f_2 - u} \neq \emptyset$, *the best weighted approximation to* f *in* H_ω *is nonunique and* $\|u - l\|_w = 2\theta$.

Proof. By virtue of Theorems 1 and 2, Remark 1, Corollary 2 and Lemma 4 it is sufficient to prove that

$$\theta_f = \inf_{g \in H_\omega} \|f - g\|_w = \theta.$$

Since $g \in H_\omega$ implies that $g(x) - g(y) \leqslant \omega(|x - y|)$ then

$$\begin{aligned} f(x) - f(y) - \omega(|x - y|) &\leqslant f(x) - f(y) + g(y) - g(x) \\ &\leqslant \|f - g\|_w \, (1/w(x) + 1/w(y)) \end{aligned}$$

for all $(x, y) \in X \times X$. Hence $\theta_f \geqslant \theta$. On the other hand, for $l \in H_\omega$ defined by (25) and all $x \in X$ we have

$$w(x)[f(x) - l(x)] \leqslant w(x)[f(x) - f_1(x)] = \theta$$

and either

$$w(x)[l(x) - f(x)] = w(x)[f_1(x) - f(x)] = -\theta \leqslant \theta$$

or

$$\begin{aligned} w(x)[l(x) - f(x)] &\leqslant w(x)[f_1(t) - \omega(|x - t|) - f(x) + \varepsilon] \\ &= w(x)[f(t) - f(x) - \omega(|x - t|) - \theta/w(t) + \varepsilon] \\ &\leqslant w(x) \Bigg[f(t) - f(x) - \omega(|x - t|) \\ &\qquad - \frac{w(x)}{w(t) + w(x)} (f(t) - f(x) - \omega(|x - t|)) + \varepsilon \Bigg] \\ &= w(x) \left[\frac{w(t)}{w(t) + w(x)} (f(t) - f(x) - \omega(|x - t|)) + \varepsilon \right] \\ &\leqslant \theta + \varepsilon \, \|w\|, \end{aligned}$$

where $\varepsilon > 0$ is arbitrary and t dependent on ε is such that $l(x) \leqslant f_1(t) - \omega(|x - t|) + \varepsilon$. This gives $\|f - l\|_w \leqslant \theta$. Consequently, we obtain $\theta_f = \theta$. This completes the proof. ∎

Let us now assume that $X=[a,b]$ is a compact interval of the real line and that $\omega \in C[0, b-a]$ is a modulus of continuity, i.e., that $\lim_{t\to 0+} \omega(t) = \omega(0)=0$, $0 \leqslant \omega(y)-\omega(x) \leqslant \omega(y-x)$ for all x, y $(0 \leqslant x \leqslant y \leqslant b-a)$. Then the sets $H_\omega \cap C[a,b]$ coincide with the well known sets H^ω (see, e.g. [3, p. 183]). In particular, if $\omega(x)=c\cdot x^\alpha$ $(0\leqslant x \leqslant b-a$ and $0<\alpha\leqslant 1)$ then $H_\omega \cap C[a,b]$ contains all functions from $C[a,b]$ which satisfy the Hölder condition with constant c. From Theorem 5 we obtain the following corollary.

COROLLARY 3. *Let ω be a modulus of continuity, $w \in C[a,b]$ and let $G=H_\omega \cap C[a,b]$. Then for each $f \in C[a,b]\backslash G$ there exists a best weighted approximation to f in G, the set of all best weighted approximations G_f is equal to $[l,u]\cap G$ and the error θ_f is equal to θ, where $l, u \in G_f$ and θ are given by formulae* (25). *Moreover, the error determining set N_f is given by the same formula as in Theorem* 5, *$\|u-l\|_w \leqslant 2\theta$, and $E_f \neq \varnothing$ implies that $\|u-l\|_w = 2\theta$.*

5. APPROXIMATION BY EVEN FUNCTIONS

In this section let s: $X \to X$ be a one–one map of an abstract set X on itself. Define

$$R_s = \{g \in B(X)\colon g(s(x)) = g(x) \text{ for each } x \in X\}.$$

In particular, if $s(x)=-x$ on a subset $X \subset \mathbb{R}$ such that $x \in X$ implies $-x \in X$, then R_s is a set of all even functions on X.

THEOREM 6. *Let $f \in B(X)\backslash R_s$. Then there exists a best weighted approximation to f in R_s, the set of all best weighted approximations to f in R_s is equal to $[l,u]\cap R_s$, and the error*

$$\theta_f = \inf_{g\in R_s} \|f-g\|_w$$

is equal to θ, where

$$\begin{gathered}\theta = \sup_{x\in X} \frac{w(x)\, w(s(x))}{w(x)+w(s(x))}\,[f(s(x))-f(x)],\\ l(x)=\max[f_1(x), f_1(s(x))], \qquad u(x)=\min[f_2(x), f_2(s(x))],\\ f_1 = f-\theta/w \qquad \text{and} \qquad f_2 = f+\theta/w.\end{gathered} \tag{26}$$

Moreover, the error determining set is equal to

$$N_f = (Z_{f_1 - l} \cap Z_{f_1 - u}) \cup (Z_{f_2 - l} \cap Z_{f_2 - u}).$$

In the case when there exists $x_0 \in X$ such that $s(x_0) = x_0$, the best weighted approximation is nonunique and $\|u - l\|_w = 2\theta$.

Proof. Obviously, $l, u \in R_s$. If $g \in R_s$ and $g \geqslant f_1$ then from $g(x) \geqslant f_1(x)$, $g(s(x)) \geqslant f_1(s(x))$ and $g(x) = g(s(x))$ for all $x \in X$ we immediately obtain $g(x) \geqslant l(x)$ for all $x \in X$. Hence condition (i) in Definition 2 is satisfied for R_s. Similarly, we may verify condition (ii) in Definition 2. Since condition (iii) from Definition 2 also holds for the set R_s, then the set R_s is admissible with respect to (f_1, f_2). Moreover, if s has a fixed point x_0 in X then $f_1(x_0) = l(x_0)$, $f_2(x_0) = u(x_0)$, and so $x_0 \in Z_{f_1 - l} \cap Z_{f_2 - u}$. Therefore, in view of Theorems 1 and 2, Remark 1, Corollary 2 and Lemma 4 it is sufficient to prove that $\theta_f = \theta$. Note that

$$\begin{aligned} f(s(x)) - f(x) &= f(s(x)) - g(s(x)) + g(x) - f(x) \\ &\leqslant \|f - g\|_w \, [1/w(x) + 1/w(s(x))] \end{aligned}$$

for all $g \in R_s$ and all $x \in X$. Hence $\theta_f \geqslant \theta$. On the other hand, for every $x \in X$ we have

$$w(x)[f(x) - l(x)] \leqslant w(x)[f(x) - f_1(x)] = \theta$$

and either

$$w(x)[l(x) - f(x)] = w(x)[f_1(x) - f(x)] = -\theta \leqslant \theta$$

or

$$\begin{aligned} &w(x)[l(x) - f(x)] \\ &\quad = w(x)[f_1(s(x)) - f(x)] \\ &\quad = w(x)[f(s(x)) - \theta/w(s(x)) - f(x)] \\ &\quad \leqslant w(x)\left(f(s(x)) - \frac{w(x)}{w(x) + w(s(x))} [f(s(x)) - f(x)] - f(x)\right) \\ &\quad = \frac{w(x)\, w(s(x))}{w(x) + w(s(x))} [f(s(x)) - f(x)] \leqslant \theta. \end{aligned}$$

Hence $\|f - l\|_w \leqslant \theta$. Thus $\theta_f = \theta$. This completes the proof. ∎

If s does not have a fixed point in X, then a best weighted approximation can either be unique or nonunique. Indeed, if $s(x) = -x$, $w(x) = 1$, and $f(x) = x^3$ on $X = [-2, -1] \cup [1, 2]$ then $f_1(x) = -|x|^3$, $f_2(x) = |x|^3$, $\theta = 8$,

$l(x)=|x|^3-8$ and $u(x)=-|x|^3+8$. Hence by Theorem 6 the best approximation is not unique. But in the case $X=[-2,-1]\cup[1,2]$ and

$$\begin{aligned} f(x) &= -x-1, \qquad x\in[-2,-1] \\ &= x, \qquad\qquad x\in[1,2] \end{aligned}$$

we have $f_1(x)=|x|-1$, $f_2(x)=|x|$, $\theta=\frac{1}{2}$, $l(x)=u(x)=|x|-\frac{1}{2}$, i.e., by Theorem 6 the best approximation is unique.

Remark 2. If we additionally assume that X is a topological space and that $f, w\in C_b(X)$, then from the formulae on l and u given in (26) it follows that we may replace $B(X)$ by $C_b(X)$ in Theorem 6 without the loss of its validity.

Acknowledgment

The author wishes to thank the referee for helpful suggestions and for the above proofs of Theorem 1 and Corollary 1.

References

1. G. Birkhoff, "Lattice Theory," Amer. Math. Soc., New York, 1948.
2. C. B. Dunham, Chebyshev approximation by families with the betweenness property, *Trans. Amer. Math. Soc.* **136** (1969), 151–157.
3. N. P. Korneicuk, "Optimization Problems in Approximation Theory," Moscow, 1976.
4. R. Smarzewski, Some remarks on nonlinear Chebyshev approximation to functions defined on normal space, *J. Approx. Theory* **24** (1978), 169–175.
5. V. A. Ubhaya, Isotone optimization, I, *J. Approx. Theory* **12** (1974), 146–159.
6. V. A. Ubhaya, Isotone optimization, II, *J. Approx. Theory* **12** (1974), 315–331.
7. V. A. Ubhaya, Almost monotone approximation in L_{∞}, *J. Math. Anal. Appl.* **49** (1975), 659–679.